On the Logic of Merging: Quota and Gmin Merging Operators

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Abstract

In this paper, two families of merging operators are considered: quota operators and *Gmin* operators. Quota operators rely on a simple idea: any possible world is viewed as a model of the result of the merging when it satisfies "sufficiently many" bases from the given profile (a multi-set of bases). Different interpretations of the "sufficiently many" give rise to specific operators. Each *Gmin* operator is parameterized by a pseudo-distance and each of them is intended to refine the quota operators (i.e., to preserve more information). Quota and *Gmin* operators are evaluated and compared along four dimensions: rationality, computational complexity, strategy-proofness, and discriminating power. Those two families are shown as interesting alternatives to the formula-based merging operators (which selects some formulas in the union of the bases).

1 Introduction

Merging operators aim at defining the beliefs (resp. goals) of a group of agents from their individual beliefs (resp. goals). The merging problem in the propositional setting has been addressed in many works, both from the artificial intelligence community and the database community [6; 23; 18; 19; 3; 4; 16]. It is also close to important issues considered in Social Choice Theory [1; 21; 2].

Each operator is more or less suited to the many merging scenarios which can be considered. Subsequently, when facing an application for which merging is required, a first difficulty is the choice of a specific merging operator. Among the criteria which can be used to make a clever choice, are the following ones:

Rationality: A main requirement for adhering to a merging method is that it offers the expected properties of what intuitively "merging" means. This calls for sets of rationality postulates and this has been addressed in several papers [23; 18; 16]. In the following, we focus on the rationality postulates given in [16], because they extend other proposals. The more (w.r.t. set inclusion) postulates satisfied the more rational the operator.

Computational Complexity: When one looks for a merging operator for an autonomous multi-agent system, a natural requirement is computational efficiency. In the worst case, merging is not a computationally easy task, and query answering typically lies at the first or even the second level of the polynomial hierarchy. Computationally easier operators can be obviously preferred to more complex ones.

Strategy-proofness: It is usually expected for merging that agents report truthfully their beliefs/goals. For many applications, this assumption can easily be made, in particular when the agents have limited reasoning abilities. However, when rational agents with full inference power are considered, such an assumption must be questioned: agents can be tempted to misreport their beliefs/goals in order to achieve a better merging result from their point of view. Strategy-proof operators must be preferred in such a case.

Discriminating Power: Because information is typically hard to be acquired, another important criterion to compare merging operators is cautiousness: merging operators which preserve only few information from the individual bases cannot be considered as valuable ones. Thus, it is natural to prefer operators leading to consistent merged bases that are as strong as possible from an inferential point of view.

As to rationality, one can look at [23; 18; 19;

13; 16; 14]. As to computational complexity, see [14], and for a study of strategy-proofness of many merging operators, a recent reference is [10] (see also [20] for a related study concerning OCF merging operators). In light of those results, it appears that no merging operator is a better performer than any other operator with respect to the four criteria. To be more precise model-based operators[1] are often computationally easier (inference is typically Θ_2^p-complete or Δ_2^p-complete) than formula-based ones (inference can be Π_2^p-complete) [14]. Model-based operators also typically satisfy more rationality postulates (see [16; 13]). The last two criteria are much more difficult to satisfy for both families of operators, even in very restricted cases. Actually, most of the merging operators identified so far in the literature are not strategy-proof. Since the four evaluation criteria appear as hard to be satisfied altogether, one cannot do better than searching for good trade-offs.

We consider in this paper two families of propositional merging operators. The first one consists of quota merging operators. Quota operators rely on a simple idea: any possible world is viewed as a model of the result of the merging when it satisfies "sufficiently many" bases from the given profile. "Sufficiently many" can mean either "at least k" (any integer, absolute quota), or "at least $k\%$" (a relative quota), or finally "as many as possible", and each interpretation gives rise to a specific merging operator. We show that those operators exhibit good logical properties, have low computational complexity and are strategy-proof. Since this is achieved at the price of a potential lack of discriminating power, we introduce a second family of merging operators: *Gmin* operators. Each *Gmin* operator is parameterized by a pseudo-distance and each of them is intended to refine the quota operators (i.e., to preserve more information). Such operators are both more rational and more discriminating than quota merging operators. Unfortunately, this improvement has to be paid by a higher computational complexity, and more strategic vulnerabilities, but we think they offer an interesting compromise nevertheless.

Note that aggregation functions close to the ones on which quota and *Gmin* operators are based are used to deal with relational structures that are more complex than bipartitions of worlds (which are the structures under consideration in standard propositional logic). For instance, they have been considered in the possibilistic logic setting and for constraint satisfaction problems (see e.g. [9; 8; 11]). However, as far as we know, no systematic study of quota and *Gmin* operators has been conducted so far in the standard propositional setting. Especially, they have never been evaluated with respect to the four criteria we consider. This is where the main contribution of the paper lies.

The rest of the paper is as follows. The next section gives some notations and definitions. In Section 3, quota operators are defined and their properties are studied. In Section 4, we define $\triangle^{k_{\max}}$, which is the operator obtained when optimizing the value of the quota under the constraint that it does not lead to an inconsistent merged base. In Section 5, $\triangle^{\text{GMIN}}$ operators are defined and their properties are studied. Finally, we conclude this paper in Section 6. Proofs are omitted due to space limitations.

2 Formal Preliminaries

We consider a propositional language $\mathcal{L}$ defined from a finite set of propositional variables $\mathcal{P}$ and the usual connectives, including $\top$ (the boolean constant always true) and $\bot$ (the boolean constant always false).

An interpretation (or world) is a total function from $\mathcal{P}$ to $\{0,1\}$, denoted by a bit vector whenever a strict total order on $\mathcal{P}$ is specified. The set of all interpretations is noted $\mathcal{W}$. An interpretation ω is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $[\phi]$ denotes the set of models of formula ϕ, i.e., $[\phi] = \{\omega \in \mathcal{W} \mid \omega \models \phi\}$.

A *base* K denotes the set of beliefs/goals of an agent, it is a finite and consistent set of propositional formulas, interpreted conjunctively. Unless stated otherwise, we identify K with the conjunction of its elements.

A *profile* E denotes the group of agents that is involved in the merging process. It is a multi-set (bag) of belief/goal bases $E = \{K_1, \ldots, K_n\}$ (hence two agents are allowed to exhibit identical bases). We denote by $\bigwedge E$ the conjunction of bases of E, i.e., $\bigwedge E = K_1 \wedge \ldots \wedge K_n$, and $\bigvee E$ is the disjunction of the bases of E, i.e., $\bigvee E = K_1 \vee \ldots \vee K_n$. A profile E is said to be consistent if and only if $\bigwedge E$ is consistent. The multi-set union is noted $\sqcup$, multi-set containment relation is noted $\sqsubseteq$. The cardinal of a finite set (or a finite multi-set) A is noted $\#(A)$. We say that two profiles are equivalent, noted $E_1 \equiv E_2$, if there exists a bijection f from E_1 to E_2 such that for every $\phi \in E_1$, ϕ and $f(\phi)$ are logically equivalent.

The result of the merging of the bases of a profile E, under the integrity constraints μ is the *merged base* denoted $\triangle_\mu(E)$. The *integrity constraints* consist of a consistent formula the merged base has to satisfy (it may represent some physical laws, some norms, etc.).

We assume the reader familiar with the following classes located at the first level of the poly-

[1] A distinction between model-based operators, which select some interpretations that are the "closest" to the bases, and formula-based ones, which pick some formulas in the union of the bases is often made [14].

nomial hierarchy (see [22] for an introduction to complexity theory): BH_2, $\Delta_2^p = \mathsf{P}^{\mathsf{NP}}$ and $\Theta_2^p = \Delta_2^p[\mathcal{O}(log\, n)]$.

3 Quota Operators

Let us first define the quota operators.

Definition 1 *Let k be an integer, $E = \{K_1, \ldots, K_n\}$ be a profile, and μ be a formula. The k-*quota merging operator*, denoted $\triangle^k$, is defined in a model-theoretic way as:*
$[\triangle_\mu^k(E)] =$

$$\begin{cases} \{\omega \in [\mu] \mid \forall K_i \in E\ \omega \models K_i\} \\ \qquad\qquad \textit{if non empty,} \\ \{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\} \\ \qquad\qquad \textit{otherwise.} \end{cases}$$

Essentially, this definition states that the models of the result of the k-quota merging of profile E under constraints μ are the models of μ which satisfy at least k bases of E. When there is no conflict for the merging, i.e., $\bigwedge E \wedge \mu$ is consistent, the result of the merging is simply the conjunction of the bases.

Example 1 *Let us consider the following example, with a profile $E = \{K_1, K_2, K_3, K_4\}$, such that $[K_1] = \{100, 001, 101\}$, $[K_2] = \{001, 101\}$, $[K_3] = \{100, 000\}$, and $[K_4] = \{111\}$, and the integrity constraints $[\mu] = \mathcal{W} \setminus \{010, 011\}$. With quota operators we get as a result $[\triangle_\mu^1(E)] = \{000, 001, 100, 101, 111\}$, $[\triangle_\mu^2(E)] = \{001, 100, 101\}$ and $[\triangle_\mu^3(E)] = \emptyset$.*

Here is an equivalent syntactical characterization of $[\triangle_\mu^k(E)]$ (i.e., the result is directly given by a formula) that is obtained from preferred consistent subsets of E.[2] Let us first define the following notation:

$$\ulcorner n_k \urcorner = \{C \subseteq \{1, \ldots, n\} \mid \#(C) = k\}.$$

Then the following proposition gives a characterization of quota operators :

Proposition 1 *Let k be an integer, $E = \{K_1, \ldots, K_n\}$ be a profile, and μ be a formula.*

$$\triangle_\mu^k(E) \equiv \begin{cases} \bigwedge E \wedge \mu & \textit{if consistent,} \\ (\bigvee_{C \in \ulcorner n_k \urcorner} (\bigwedge_{j \in C} K_j)) \wedge \mu & \textit{otherwise.} \end{cases}$$

Interestingly, the size of the formula equivalent to $[\triangle_\mu^k(E)]$ given by Proposition 1 is polynomial in $|E| + |\mu|$. Hence, merged bases can be easily compiled as propositional formulas.

[2] To be more precise, "subsets" stands here for multi-set containment.

3.1 Logical Properties

Since we aim at investigating the logical properties of our family of merging operators, a set of properties must first be considered as a base line. The following set of postulates was pointed out in [15; 16]:

Definition 2 (IC merging operators) $\triangle$ *is an* IC merging operator *if and only if it satisfies the following postulates:*

(IC0) $\triangle_\mu(E) \models \mu$

(IC1) *If μ is consistent, then $\triangle_\mu(E)$ is consistent*

(IC2) *If $\bigwedge E$ is consistent with μ, then $\triangle_\mu(E) \equiv \bigwedge E \wedge \mu$*

(IC3) *If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\triangle_{\mu_1}(E_1) \equiv \triangle_{\mu_2}(E_2)$*

(IC4) *If $K_1 \models \mu$ and $K_2 \models \mu$, then $\triangle_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\triangle_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent*

(IC5) $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2) \models \triangle_\mu(E_1 \sqcup E_2)$

(IC6) *If $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ is consistent, then $\triangle_\mu(E_1 \sqcup E_2) \models \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$*

(IC7) $\triangle_{\mu_1}(E) \wedge \mu_2 \models \triangle_{\mu_1 \wedge \mu_2}(E)$

(IC8) *If $\triangle_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(E) \models \triangle_{\mu_1}(E)$*

An IC merging operator is said to be an IC majority operator *if it satisfies* **(Maj)**

(Maj) $\exists n\ \triangle_\mu(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}) \models \triangle_\mu(E_2)$

Quota merging operators exhibit good logical properties.

Proposition 2 $\triangle^k$ *operators satisfy properties* **(IC0)**, **(IC2)**, **(IC3)**, **(IC4)**, **(IC5)**, **(IC7)** *and* **(IC8)**. *They do not satisfy* **(IC1)**, **(IC6)** *and* **(Maj)** *in general.*

Only two properties of IC merging operators are not satisfied: **(IC1)**[3] since the result of the quota merging can be inconsistent (see Example 1), and **(IC6)**.

Beside those general properties, some specific additional properties, are satisfied by quota operators.

(Disj) If $(\bigvee E) \wedge \mu$ is consistent, then $\triangle_\mu(E) \models (\bigvee E) \wedge \mu$

[3] It is possible to make **(IC1)** satisfied by requiring that, when no interpretation reaches the quota (i.e., satisfies at least k bases), the merged base is equivalent to the integrity constraints formula. But this definition leads to operators which do not satisfy **(IC5)**. This last property is very important from an aggregation point of view. It corresponds to a Pareto condition, that is considered as a minimal rationality requirement for aggregation in Social Choice Theory [1; 21; 2]. This is why we do not consider such an additional family of operators any longer in the following.

Interestingly, the disjunction property **(Disj)** is not shared by every IC majority merging operator [16], since most of them allow for "generating" some new beliefs/goals from the ones in the bases of the profile (some interpretations that do not satisfy any of the bases can be chosen as models of the merged base). When this behaviour is unexpected, formula-based merging operators – which satisfy **(Disj)** – can be used, but such operators do not satisfy many rationality postulates [13] (especially **(IC3)** is not satisfied) and are often hard from a computational point of view. Quota operators (as well as the other operators studied in this paper) which also ensure **(Disj)** offer interesting alternatives to formula-based operators in this respect.

Two other interesting postulates can be defined for characterizing more precisely quota operators; the first one is a weakening of **(Maj)**, which is not satisfied by every IC merging operator:

(Wmaj) If $\triangle_\mu(E_2)$ is consistent, then $\exists n\ \triangle_\mu\ (E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}) \wedge \triangle_\mu(E_2)$ is consistent

The second one shows the prominence of the largest maximal consistent subsets of the profile; let us define $Maxcons_\mu(E)$ as $\{M \mid M \sqsubseteq E, \bigwedge M \wedge \mu$ is consistent, and $\forall M' M \sqsubset M' \sqsubseteq E, \bigwedge M' \wedge \mu$ is not consistent$\}$:

(Card) If $E_1, E_2 \in Maxcons_\mu(E)$, $\#(E_1) \leq \#(E_2)$, and $\triangle_\mu(E) \wedge E_1$ is consistent, then $\triangle_\mu(E) \wedge E_2$ is consistent

Proposition 3 $\triangle^k$ *operators satisfy properties* **(Disj)**, **(Card)** *and* **(Wmaj)**.

Note that it is not the case that every IC majority merging operator satisfies **(Card)** (see Section 5).

3.2 Computational Complexity

Let $\triangle$ be a merging operator, we consider the following decision problem MERGE($\triangle$):

- **Input :** a triple $\langle E, \mu, \alpha \rangle$ where $E = \{K_1, \ldots, K_n\}$ is a profile, $\mu \in \mathcal{L}$ is a formula, and $\alpha \in \mathcal{L}$ is a formula.
- **Question :** Does $\triangle_\mu(E) \models \alpha$ hold?

For quota merging operators, we can prove that:

Proposition 4 MERGE($\triangle^k$) *is* BH(2)*-complete.*

This BH(2)-completeness result is obtained even in the restricted case the query is a propositional symbol and there is no integrity constraints ($\mu = \top$). Note that this complexity class is located at a low level of the boolean hierarchy. And that, obviously, the complexity of MERGE($\triangle^k$) decreases to coNP whenever k is not lower than the number of bases of E (or under the restriction when $\bigwedge E \wedge \mu$ is inconsistent).

3.3 Strategy-Proofness

Let us now investigate how robust quota operators are with respect to manipulation. Intuitively, a merging operator is strategy-proof if and only if, given the beliefs/goals of the other agents, reporting untruthful beliefs/goals does not enable an agent to improve her satisfaction. A formal counterpart of it is given in [10]:

Definition 3 (strategy-proofness) *Let i be a satisfaction index, i.e., a total function from $\mathcal{L} \times \mathcal{L}$ to $I\!R$. A merging operator Δ is* strategy-proof for *i if and only if there is no integrity constraint μ, no profile $E = \{K_1, \ldots, K_n\}$, no base K and no base K' such that $i(K, \Delta_\mu(E \sqcup \{K'\})) > i(K, \Delta_\mu(E \sqcup \{K\}))$.*

Clearly, there are numerous different ways to define the satisfaction of an agent given a merged base. While many *ad hoc* definitions can be considered, the following three indexes from [10] are meaningful when no additional information are available:

Definition 4 (indexes)

- $i_{d_w}(K, K_\Delta) = \begin{cases} 1 & \textit{if } K \wedge K_\Delta \textit{ is consistent,} \\ 0 & \textit{otherwise.} \end{cases}$
- $i_{d_s}(K, K_\Delta) = \begin{cases} 1 & \textit{if } K_\Delta \models K, \\ 0 & \textit{otherwise.} \end{cases}$
- $i_p(K, K_\Delta) = \begin{cases} \frac{\#([K] \cap [K_\Delta])}{\#([K_\Delta])} & \textit{if} \#([K_\Delta]) \neq 0, \\ 0 & \textit{otherwise.} \end{cases}$

For the weak drastic index (i_{d_w}), the agent is considered fully satisfied as soon as its beliefs/goals are consistent with the merged base. For the strong drastic index (i_{d_s}), in order to be fully satisfied, the agent must impose her beliefs/goals to the whole group. The last index ("probabilistic index" i_p) is not a boolean one, leading to a more gradual notion of satisfaction. The more compatible the merged base with the agent's base the more satisfied the agent. The compatibility degree of K with K_Δ is the (normalized) number of models of K that are models of K_Δ as well.

Interestingly, we can prove that:

Proposition 5 *Quota merging operators are strategy-proof for i_p, i_{d_w} and i_{d_s}.*

This is quite noticeable since strategy-proof merging operators are not numerous [10]. Strategy-proofness is hard to achieve, as illustrated in Social Choice Theory, for aggregation of preference relations, by the Gibbard-Satterthwaite impossibility theorem [12; 24; 21].

3.4 Absolute and Relative Quotas

In the definition of quota merging operators, an absolute threshold, i.e., a fixed integer not depending on the number of bases in the profile, has been used. On the other hand, it can prove also sensible to express quota in a relative manner, and to define the models of the merged base as the interpretations satisfying at least half (or the two third, or the wanted ratio) of the initial bases. This technique is close to a well-known voting method used in Social Choice Theory, namely voting in committees [5]. Let us call such operators k-ratio merging operators (with $0 \leq k \leq 1$), and let us note them $\triangle^{\overline{k}}$.

Example 2 *(continued)* $[\triangle_\mu^{\overline{0.2}}(E)] = \{001, 100, 101\}$, $[\triangle_\mu^{\overline{0.3}}(E)] = [\triangle_\mu^{\overline{0.5}}(E)] = \{001, 100, 101\}$.

One can quickly figure out the close connections between the two families of quota merging operators (the one based on absolute quota and the other one on relative quota, or ratio). Each ratio merging operator corresponds to a family of quota merging operators (one for each possible cardinal of the profile). And given a fixed cardinal, each (absolute) quota merging operator corresponds to a family of ratio merging operators.

Although the intuitive motivations of the two definitions of those families look different, it turns out that ratio merging operators have exactly the same properties w.r.t. computational complexity and strategy-proofness as (absolute) quota merging operators (though the proofs of some results are different). Only some logical properties are different.

Proposition 6 $\triangle^{\overline{k}}$ *operators satisfy properties* **(IC0)**, **(IC2)**, **(IC3)**, **(IC4)**, **(IC5)**, **(IC7)**, **(IC8)**, **(Maj)**, **(Disj)** *and* **(Card)**. *They do not satisfy* **(IC1)** *and* **(IC6)** *in general.*

4 $\triangle^{k_{\max}}$ Operator

Now, whatever the chosen quota is absolute or not, an important point is the choice of its value. Let us first observe that quota merging operators lead to a sequence of merged bases that is monotonic w.r.t. logical entailment:

Proposition 7 *Let E be a profile, μ be a formula. We have $\triangle_\mu^{k+1}(E) \models \triangle_\mu^k(E)$ for all integers k.*

Among the elements of this sequence, some of them are of special interest. Thus, $\triangle^0$ gives the conjunction of the bases (with the constraints) when consistent and μ otherwise. It is called *full meet merging operator* in [15]. $\triangle^1$ gives the conjunction of the bases (with the constraints) when consistent and the disjunction of the bases (with the constraints) otherwise; it is closed to the *basic merging operator* [15], and is also definable as a model-based merging operator obtained using the drastic distance and $\max$ as aggregation function [14]. The only difference is that $\triangle^1$ gives an inconsistent result when the disjunction of the bases is not consistent with the constraints, whilst the basic merging operator gives μ in this case.

Each time k is increased, the result of the merging is either the same as for the previous value of k or is logically stronger. In our finite propositional framework, the sequence $(\triangle_\mu^k(E))(k > 0)$ is obviously stationary from some stage. The value for which it becomes stationary is not interesting in itself, since the corresponding merged base is either equivalent to the conjunction of the bases of the profile (with the constraints), or to the inconsistent base. But an interesting value of k is the one leading to the last nontrivial merged base.

Definition 5 *Let $E = \{K_1, \ldots, K_n\}$ be a profile, μ be a formula. Let $k_{\max} = \max(\{i \leq \#(E) \mid \triangle_\mu^i(E) \not\models \bot\})$. $\triangle^{k_{max}}$ is defined in a model-theoretic way as:*

$$[\triangle_\mu^{k_{max}}(E)] = \begin{cases} \{\omega \in [\mu] \mid \forall K_i \in E\ \omega \models K_i\} & \textit{if non empty,} \\ \{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) = k_{\max}\} & \textit{otherwise.} \end{cases}$$

While very close to quota operators, the resulting operator $\triangle^{k_{\max}}$ is not a true quota operator since the value of k_{max} is not given a priori, but depends on E and μ.

Example 3 *(continued)* $[\triangle_\mu^{k_{\max}}(E)] = \{001, 100, 101\}$.

At a first glance, $\triangle^{k_{\max}}$ looks similar to the formula-based operator Δ^{C4} which selects cardinality-maximal subbases in the union of the bases from the profile [13; 3; 4]; however, $\triangle^{k_{\max}}$ and Δ^{C4} are distinct; thus, while both operators satisfy **(Disj)**, $\triangle^{k_{\max}}$ satisfies **(IC3)** and **(Maj)** while Δ^{C4} satisfies none of them. Contrastingly, $\triangle^{k_{\max}}$ belongs to two important families of model-based merging operators, namely the $\triangle^{\Sigma}$ family and the $\triangle^{GMax}$ family when the drastic distance is used [17]. Accordingly, $\triangle^{k_{\max}}$ has very good logical properties:

Proposition 8 $\triangle^{k_{\max}}$ *satisfies* **(IC0 - IC8)**, **(Maj)**, **(Disj)** *and* **(Card)**.

Clearly enough, $\triangle^{k_{\max}}$ is obtained by considering the problem of optimizing the quota (for "pure" quota operators, k is given, so it does not need to be computed). Unsurprisingly, the corresponding inference problem is computationally harder than the inference problem for quota operators (under the standard assumptions of complexity theory):

Proposition 9 MERGE($\triangle^{k_{\max}}$) *is* Θ_2^p*-complete.*

Clearly enough, if k_{max} is computed during an off-line pre-processing stage and becomes part of the input afterwards, the complexity falls down to coNP.

Now, as to strategy-proofness, the $k_{\max}$ operator exhibits all the good properties of quota operators.

Proposition 10 $\triangle^{k_{\max}}$ *is strategy-proof for the three indexes* i_p, i_{d_w} *and* i_{d_s}.

5 $\triangle^{\text{GMIN}}$ Operators

Starting from $\triangle^{k_{\max}}$, one could wonder whether it is possible to constrain further the quota operators so as to get operators with a higher discriminating power, i.e,. allowing more inferences to be drawn. In this section we provide a full family of such operators.

In order to define a $\triangle^{\text{GMIN}}$ operator, the definition of a pseudo-distance between interpretations is first needed:

Definition 6 *A pseudo-distance between interpretations is a function* d *from* $\mathcal{W} \times \mathcal{W}$ *to* $\mathbb{N}$ *such that for every* $\omega_1, \omega_2 \in \mathcal{W}$

- $d(\omega_1, \omega_2) = d(\omega_2, \omega_1)$, *and*
- $d(\omega_1, \omega_2) = 0$ *if and only if* $\omega_1 = \omega_2$.

Any pseudo-distance between interpretations d *induces a "distance" between an interpretation* ω *and a formula* K *given by* $d(\omega, K) = \min_{\omega' \models K} d(\omega, \omega')$.

Examples of some such distances are the *drastic distance*, noted d_D, that gives 0 when $\omega_1 = \omega_2$ and 1 otherwise, or the *Dalal distance* [7], noted d_H, that is the Hamming distance between interpretations.

Then $\triangle_{\mu}^{d,\text{GMIN}}$ operators are defined as:

Definition 7 *Let* d *be a pseudo-distance,* μ *an integrity constraint,* $E = \{K_1, \ldots, K_n\}$ *a profile and let* ω *be an interpretation. The "distance"* $d_{d,Gmin}(\omega, E)$ *is defined as the list of numbers* $(d_1, \ldots, d_n)$ *obtained by sorting in increasing order the set* $\{d(\omega, K_i) \mid K_i \in E\}$*. The models of* $\triangle_{\mu}^{d,\text{GMIN}}(E)$ *are the models of* μ *that are minimal w.r.t. the lexicographic order induced by the natural order.*

Example 4 *(continued)* $[\triangle_{\mu}^{d_D,\text{GMIN}}(E)] = \{001, 100, 101\}$. $[\triangle_{\mu}^{d_H,\text{GMIN}}(E)] = \{101\}$. *The computations are reported in Table 1. Each* $\mathbf{K_i}$ *column gives the "distance"* $d_H(\omega, K_i)$ *between the models of the integrity constraints and* K_i.

Clearly enough, $\triangle^{k_{\max}}$ is a specific *Gmin* operator:

Proposition 11 $\triangle^{d_D,\text{GMIN}} = \Delta^{k_{\max}}$.

ω	$\mathbf{K_1}$	$\mathbf{K_2}$	$\mathbf{K_3}$	$\mathbf{K_4}$	$\mathbf{d_{d_H,Gmin}}(\omega, \mathbf{E})$
000	1	1	0	3	(0,1,1,3)
001	0	0	1	2	(0,0,1,2)
100	0	1	0	2	(0,0,1,2)
101	0	0	1	1	(0,0,1,1)
110	1	2	1	1	(1,1,1,2)
111	1	1	2	0	(0,1,1,2)

Table 1: $\triangle^{d_H,\text{GMIN}}$ operator.

As far as discriminating power is concerned, $\triangle^{\text{GMIN}}$ operators are interesting operators, since they refine the operator $\Delta^{k_{\max}}$ (so they refine also every quota merging operator), as stated by the following property:

Proposition 12 *For any pseudo-distance* d*, any integrity constraint* μ *and any profile* E*, we have* $\triangle_{\mu}^{d,\text{GMIN}}(E) \models \Delta_{\mu}^{k_{\max}}(E)$.

Furthermore, *Gmin* operators exhibit very good logical properties:

Proposition 13 *Let* d *be any pseudo-distance,* $\triangle^{d,\text{GMIN}}$ *satisfies* **(IC0 - IC8)**, **(Maj)** *and* **(Disj)**. *It does not satisfy* **(Card)** *in general.*

Thus, like formula-based merging operators, $\triangle^{\text{GMIN}}$ operators satisfy (**Disj**), but contrariwise to formula-based merging operators, $\triangle^{\text{GMIN}}$ operators are IC merging operators.

Let us now investigate the strategy-proofness issue for the $\triangle^{\text{GMIN}}$ operators. In the general case, strategy-proofness of quota merging operators is lost:

Proposition 14 *Let* d *be a pseudo-distance,* $\triangle^{d,\text{GMIN}}$ *is not strategy-proof for any index among the three indexes* i_{d_w}, i_p *and* i_{d_s}.

We can guarantee strategy-proofness, but only in some very specific cases:

Proposition 15

- $\triangle^{d,\text{GMIN}}$ *is strategy-proof for* i_p, i_{d_w} *and* i_{d_s} *if the bases are complete (i.e. each base has a unique model),*
- $\triangle^{d,\text{GMIN}}$ *is strategy-proof for the indexes* i_{d_w} *and* i_{d_s} *when* $\#(E) = 2$ *and* $\mu = \top$.

Finally, let us turn to the computational complexity criterion. The next proposition is a direct consequence of a result from [14]:

Proposition 16 *Assume that the pseudo-distance* d *of any pair of interpretations* ω_1 *and* ω_2 *can be computed in time polynomial in* $|\omega_1| + |\omega_2|$*. Then* MERGE($\triangle^{d,\text{GMIN}}$) *is in* Δ_2^p.

For specific choices of d, more precise results can be derived:

Proposition 17

- MERGE($\triangle^{d_D,\textsc{Gmin}}$) *is* Θ_2^p*-complete.*
- MERGE($\triangle^{d_H,\textsc{Gmin}}$) *is* Δ_2^p*-complete.*

As expected, the complexity of inference for $\triangle^{d,\textsc{Gmin}}$ operators is higher than the complexity of inference for quota operators (under the usual assumptions of complexity theory). However, it remains at the first level of the polynomial hierarchy under reasonable requirements on the pseudo-distance.

6 Conclusion

We have considered two families of merging operators, and investigated the properties of their operators with respect to four criteria: rationality, computational complexity, strategy-proofness and discrimating power. We claim that those four criteria are the main dimensions along with propositional merging operators have to be evaluated.

While no merging operators optimizing every criteria exist, we claim that both quota and *Gmin* operators are interesting trade-offs; even if they are not fully rational and discriminating, quota operators exhibit "low complexity" and are strategy-proof; on the other hand, *Gmin* operators are slightly more complex and not strategy-proof in the general case, but they are fully rational and much less cautious. They also lead to merged bases implying the disjunction of the bases from the considered profile, thus offering an interesting alternative to formula-based merging operators [3; 4; 13; 14], which are typically at least as hard from the complexity point of view and satisfy less rationality postulates.

This work calls for several perspectives. One of them consists in determining to what extent the four criteria used are (in)dependent. This would allow for drawing a multi-dimensional map on which merging methods could be located.

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